Galerkin Solution for the Thin Circular Iris in a TE_{11}-Mode Circular Waveguide

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Abstract—An integral equation for the transverse electric field in the aperture of a concentric circular iris in a transverse plane of a circular waveguide is approximately solved by Galerkin's method. The aperture field is represented by a finite sum of normal TE and TM circular waveguide modes that fit the circular aperture. The numerical convergence of the Galerkin solution is demonstrated via resultant aperture field distributions and equivalent shunt susceptance for the case of dominant TE_{11}-mode excitation. The resultant aperture electric field distribution closely resembles that of the TE_{11} aperture mode alone, except for edge condition behavior at the edge of the iris. A resonant or capacitive iris is possible over a restricted range of frequencies.

I. Introduction

The specific aperture under consideration is that of an infinitesimally thin circular iris of inner radius b in a transverse plane of a circular waveguide of radius a, as shown in Fig. 1. The circular iris is concentric with the axis of the circular waveguide. The case of circularly symmetric excitation (TE_{0n}, TM_{0n} modes) of this circular iris or the related step change in waveguide diameter problem is considered by many authors, for example [1]–[5]. Marcuvitz [6, p. 243] gives the equivalent shunt susceptance for TE_{11} excitation of small apertures. Gubskii *et al.* [7] formulate the Galerkin method for TE_{mn}- and TM_{mn}-mode incidence using a basis of weighted Jacobi polynomials. Unfortunately, their results are given only for TE_{0n} and TM_{0n} excitation. The use of aperture waveguide modes as a basis in the Galerkin procedure [8], [9] yields the same set of linear equations as Wexler's modal analysis [10] and as the conservation of complex power technique of Wade and MacPhie [11]. The Galerkin method also yields the same equations as the Rayleigh–Ritz method, but without having to start from an explicit variational functional [12, p. 448]. In this paper, the effect of the entire infinite set of modes in the circular waveguide is approximated by series summation, in the manner of [13]–[15]. The relative convergence problem [16], [17] and less numerical accuracy can potentially arise when the number of modes in the waveguide region is truncated, as in the implementations of [9]–[11].

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The unknown in the boundary value problem is the aperture electric field, from which the waveguide fields are determined. The Galerkin method is formulated for the case of the transverse junction between two perfectly conducting cylindrical waveguides of general cross section. The portion of the junction plane that extends into the interior (the iris) is taken to be infinitesimally thin. The formulation is implemented for the TE_{11} excitation of the circular iris in a circular waveguide, and several resultant aperture electric field distributions are given. A family of design curves for the variation of the equivalent shunt susceptance as a function of iris size b/a and electrical size of the circular waveguide κa is also given, where $\kappa = 2\pi/\lambda$ is the wavenumber of the unbounded dielectric in the waveguide.

II. Derivation of Electric Field Integral Equation

The portion of each waveguide cross section in the transverse plane $z=0$ that does not coincide with the aperture S is shorted by a perfect electric conductor (Fig. 2). The aperture S is excited by any number of modes from waveguide (a) on the left (negative z) and is excited by any number of modes from waveguide (b) on the right (positive z). The cross sectional areas of the waveguides are denoted by S_a and S_b. An iris of finite thickness is treated as the simultaneous solution of two separated junctions [18], [19]. Waveguides (a) and (b) can contain two different, lossy dielectrics.

In the junction plane $z=0$, the transverse electric and magnetic fields of regions (a) and (b) are expressed as infinite summations of the normal transverse vector mode functions of the corresponding waveguide

$$\left.\begin{aligned} \bar{E}_a &= \sum_{m=1}^{\infty} V_m \bar{e}_m \quad &(1)\\ \bar{H}_a &= \sum_{m=1}^{\infty} I_m \bar{h}_m \quad &(2)\end{aligned}\right\} \text{ on } S_a$$

$$\left.\begin{aligned} \bar{E}_b &= \sum_{n=1}^{\infty} \hat{V}_n \hat{\bar{e}}_n \quad &(3)\\ \bar{H}_b &= \sum_{n=1}^{\infty} \hat{I}_n \hat{\bar{h}}_n \quad &(4)\end{aligned}\right\} \text{ on } S_b.$$

In waveguide (a), V_m is the unknown modal voltage, I_m is

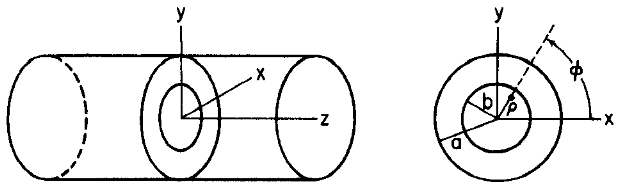

Fig. 1. Thin circular iris in circular waveguide.

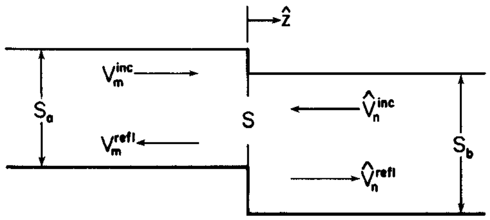

Fig. 2. Longitudinal view of transverse iris junction between two general cylindrical waveguides.

the unknown modal current, $\bar{e}_m$ is the known electric field mode vector, and $\bar{h}_m$ is the known magnetic field mode vector of the mth mode. The corresponding quantities of waveguide (b) are denoted by the caret (^) and modal index n. The entire modal spectra of the waveguide fields are needed at an abrupt discontinuity such as the junction, and so the entire infinite summations are maintained and not truncated.

It is convenient to define the inner product between two transverse vectors $\bar{A}$ and $\bar{B}$ over the surface Σ as the scalar integral

$$\langle \bar{A}, \bar{B} \rangle_\Sigma = \iint_\Sigma \bar{A} \cdot \bar{B}^* \, ds \tag{5}$$

where the asterisk denotes complex conjugate. The waveguide mode functions are orthonormalized in the sense that their pairwise inner products are given by

$$\langle \bar{e}_m, \bar{e}_k \rangle_{S_a} = \langle \bar{h}_m, \bar{h}_k \rangle_{S_a} = \delta_{mk}$$
$$\langle \hat{\bar{e}}_n, \hat{\bar{e}}_k \rangle_{S_b} = \langle \hat{\bar{h}}_n, \hat{\bar{h}}_k \rangle_{S_b} = \delta_{nk}. \tag{6}$$

The electric and magnetic vector mode functions are simply related by a 90° transverse rotation [6, p. 4]

$$\bar{h}_m = \hat{z} \times \bar{e}_m \qquad \hat{\bar{h}}_n = \hat{z} \times \hat{\bar{e}}_n \tag{7}$$

where $\hat{z}$ is th axial unit vector. The usual transmission line equations for the mth waveguide (a) mode and the nth waveguide (b) mode are

$$V_m = V_m^{\text{inc}} + V_m^{\text{refl}} \tag{8}$$
$$I_m = y_m \left[V_m^{\text{inc}} - V_m^{\text{refl}} \right] \tag{9}$$
$$\hat{V}_n = \hat{V}_n^{\text{inc}} + \hat{V}_n^{\text{refl}} \tag{10}$$
$$\hat{I}_n = -\hat{y}_n \left[\hat{V}_n^{\text{inc}} - \hat{V}_n^{\text{refl}} \right] \tag{11}$$

where the left-hand terms are total modal voltage and current. Equations (8) and (9) relate total modal voltage V_m and current I_m to the incident and reflected modal voltages V_m^{inc} and V_m^{refl} of waveguide (a) in the reference plane $z = 0$. The characteristic admittance of the mth mode of waveguide (a) is denoted by y_m, which completely accounts for the dielectric in the guide. Equations (10) and (11) give the corresponding relations of waveguide (b).

The integral equation satisfied by the unknown tangential electric field in the aperture S is obtained by enforcing continuity of the tangential magnetic field $\bar{H}$ across the aperture,

$$\sum_{m=1}^{\infty} I_m \bar{h}_m = \sum_{n=1}^{\infty} \hat{I}_n \hat{\bar{h}}_n \qquad \text{on } S. \tag{12}$$

Insertion of the electric quantities from (7)–(11) into the above, taking the vector cross product of $-\hat{z}$ with both sides, and addition of twice the incident terms to each side of the resultant equation yields

$$\sum_{m=1}^{\infty} y_m V_m \bar{e}_m + \sum_{n=1}^{\infty} \hat{y}_n \hat{V}_n \hat{\bar{e}}_n$$
$$= 2 \sum_{m=1}^{\infty} y_m V_m^{\text{inc}} \bar{e}_m + 2 \sum_{n=1}^{\infty} \hat{y}_n \hat{V}_n^{\text{inc}} \hat{\bar{e}}_n \qquad \text{on } S. \tag{13}$$

Continuity of the tangential electric field is automatically satisfied since V_m and $\hat{V}_n$ are expressed as functions of the common unknown aperture electric field in the aperture S:

$$\bar{E}_a = \sum_{m=1}^{\infty} V_m \bar{e}_m = \begin{cases} \bar{E}_{\text{aper}} & \text{on } S \\ 0 & \text{on } S_a - S \end{cases} \tag{14}$$

$$\bar{E}_b = \sum_{n=1}^{\infty} \hat{V}_n \hat{\bar{e}}_n = \begin{cases} \bar{E}_{\text{aper}} & \text{on } S \\ 0 & \text{on } S_b - S. \end{cases} \tag{15}$$

The inner product of (14) with the kth waveguide (a) electric mode vector over the waveguide (a) cross section is

$$\langle \bar{E}_a, \bar{e}_k \rangle_{S_a} = \sum_{m=1}^{\infty} V_m \langle \bar{e}_m, \bar{e}_k \rangle_{S_a} = \langle \bar{E}_{\text{aper}}, \bar{e}_k \rangle_S. \tag{16}$$

The orthogonality of the waveguide mode vectors allows each waveguide total modal voltage to be calculated separately:

$$V_m = \langle \bar{E}_{\text{aper}}, \bar{e}_m \rangle_S \qquad \hat{V}_n = \langle \bar{E}_{\text{aper}}, \hat{\bar{e}}_n \rangle_S. \tag{17}$$

Insertion of (17) into (13) yields

$$\sum_{m=1}^{\infty} y_m \bar{e}_m(\bar{\rho}) \langle \bar{E}_{\text{aper}}, \bar{e}_m \rangle_S + \sum_{n=1}^{\infty} \hat{y}_n \hat{\bar{e}}_n(\bar{\rho}) \langle \bar{E}_{\text{aper}}, \hat{\bar{e}}_n \rangle_S$$
$$= 2 \sum_{m=1}^{\infty} y_m V_m^{\text{inc}} \bar{e}_m(\bar{\rho}) + 2 \sum_{n=1}^{\infty} \hat{y}_n \hat{V}_n^{\text{inc}} \hat{\bar{e}}_n(\bar{\rho}), \qquad \bar{\rho} \in S. \tag{18}$$

This Fredholm integral equation of the first kind is cast into standard form

$$\iint_S \bar{\bar{G}}(\bar{\rho}, \bar{\rho}') \cdot \bar{E}_{\text{aper}}(\bar{\rho}') \, ds' = -2\hat{z} \times \bar{H}^{\text{inc}}(\bar{\rho}), \qquad \bar{\rho} \in S \tag{19}$$

where the dyadic Green's function is

$$\bar{\bar{G}}(\bar{\rho}, \bar{\rho}') = \sum_{m=1}^{\infty} y_m \bar{e}_m(\bar{\rho}) \bar{e}_m^*(\bar{\rho}') + \sum_{n=1}^{\infty} \hat{y}_n \hat{\bar{e}}_n(\bar{\rho}) \hat{\bar{e}}_n^*(\bar{\rho}'). \tag{20}$$

Some authors [8], [9] conceptually short the entire aperture S with a perfect electric conductor and introduce the equivalent magnetic surface current over S necessary to bring the total tangential electric field back to its correct value $\bar{E}_{\text{aper}}$. In such cases, the resultant integral equation with the magnetic surface current as the unknown is entirely equivalent to (19). The boundary condition is

$$\hat{\nu} \times \bar{E}_{\text{aper}}(\bar{\rho}) = 0, \qquad \bar{\rho} \in C \tag{21}$$

where $\hat{\nu}$ is the outward normal of the boundary curve C of the aperture S. This boundary condition must be satisfied according to (14) and (15) if the integral equation (19) is valid in S.

III. Formal Galerkin Solution of Integral Equation

The linear integral operator (19) is

$$\mathscr{L}\bar{E}_{\text{aper}}(\bar{\rho}) = \sum_{m=1}^{\infty} y_m \bar{e}_m(\bar{\rho}) \iint_S \bar{e}_m^*(\bar{\rho}') \cdot \bar{E}_{\text{aper}}(\bar{\rho}')\, ds' + \sum_{n=1}^{\infty} \hat{y}_n \hat{\bar{e}}_n(\bar{\rho}) \iint_S \hat{\bar{e}}_n^*(\bar{\rho}') \cdot \bar{E}_{\text{aper}}(\bar{\rho}')\, ds'. \tag{22}$$

The adjoint operator $\mathscr{L}^a$ satisfies [20, p. 315]

$$\langle \mathscr{L}\bar{A}, \bar{B} \rangle_S = \langle \bar{A}, \mathscr{L}^a \bar{B} \rangle_S \tag{23}$$

where $\bar{A}$ and $\bar{B}$ are elements of the domain of $\mathscr{L}$ and $\mathscr{L}^a$, respectively. Application of this definition and the Hilbert space inner product (5) gives

$$\mathscr{L}^a\bar{E}_{\text{aper}}(\bar{\rho}) = \sum_{m=1}^{\infty} y_m^* \bar{e}_m(\bar{\rho}) \iint_S \bar{e}_m^*(\bar{\rho}') \cdot \bar{E}_{\text{aper}}(\bar{\rho}')\, ds' + \sum_{n=1}^{\infty} \hat{y}_n^* \hat{\bar{e}}_n(\bar{\rho}) \iint_S \hat{\bar{e}}_n^*(\bar{\rho}') \cdot \bar{E}_{\text{aper}}(\bar{\rho}')\, ds'. \tag{24}$$

The domains of $\mathscr{L}$ and $\mathscr{L}^a$ are identical and are the set of vector functions $\bar{E}_{\text{aper}}(\bar{\rho})$ that are integrable with the waveguide vector modal functions over the aperture S and result in convergent series of (22) and (24). Furthermore, the class of legitimate aperture fields is limited to those vector functions that satisfy the boundary condition (21). The Galerkin technique is now applied in the manner of [8] and [9]. The unknown aperture electric field is approximated by a linear combination of L independent basis functions $\tilde{\bar{e}}_l(\bar{\rho})$,

$$\bar{E}_{\text{aper}}(\bar{\rho}) = \sum_{l=1}^{L} \tilde{V}_l \tilde{\bar{e}}_l(\bar{\rho}) \tag{25}$$

where the unknowns are the complex aperture voltages $\tilde{V}_l$. Inserting (25) into (18) and taking the inner product over the aperture S of the residual error with the kth basis function yields L linear equations. In matrix form these linear equations are

$$(\bar{Y}^a + \bar{Y}^b)\tilde{\bar{V}} = \bar{I}^a + \bar{I}^b \tag{26}$$

where the (k,l)th elements of the $L \times L$ square aperture admittance matrices are

$$Y_{kl}^a = \sum_{m=1}^{\infty} y_m \langle \bar{e}_m, \tilde{\bar{e}}_k \rangle_S \langle \tilde{\bar{e}}_l, \bar{e}_m \rangle_S \tag{27}$$

$$Y_{kl}^b = \sum_{n=1}^{\infty} \hat{y}_n \langle \hat{\bar{e}}_n, \tilde{\bar{e}}_k \rangle_S \langle \tilde{\bar{e}}_l, \hat{\bar{e}}_n \rangle_S \tag{28}$$

and the kth elements of the $L \times 1$ current excitation column vectors are

$$I_k^a = 2 \sum_{m=1}^{\infty} y_m V_m^{\text{inc}} \langle \bar{e}_m, \tilde{\bar{e}}_k \rangle_S \tag{29}$$

$$I_k^b = 2 \sum_{n=1}^{\infty} \hat{y}_n \hat{V}_n^{\text{inc}} \langle \hat{\bar{e}}_n, \tilde{\bar{e}}_k \rangle_S. \tag{30}$$

Note that Y_{kl}^a and I_k^a are functions only of the geometry, frequency, and excitation of the left side of the junction, and similarly for Y_{kl}^b and I_k^b on the right. This is due to the inherent separation of the Green's function (20). The coupling between waveguides (a) and (b) occurs algebraically as a simple matrix addition, with the solution obtained via matrix inversion. The element Y_{kl}^a can be thought of as the mutual admittance between the kth and lth aperture modes, where the coupling is via the infinite set of waveguide (a) modes.

IV. Modal Basis

If the aperture has the shape of a familiar waveguide cross section, then the normal TE (h-type) and TM (e-type) aperture waveguide modes constitute a candidate basis for the tangential electric field in the aperture. The transverse electric fields for the modes are expressed in terms of scalar potential wave functions [6, p. 4]

$$\text{TE modes: } \bar{e}^h = \hat{z} \times \nabla_t \Psi^h \tag{31}$$

$$\text{TM modes: } \bar{e}^e = -\nabla_t \Psi^e \tag{32}$$

which satisfy the boundary conditions

$$\frac{\partial \Psi^h}{\partial \nu} = 0 \qquad \text{on } C \text{ (Neumann or hard)} \tag{33}$$

$$\Psi^e = 0 \qquad \text{on } C \text{ (Dirichlet or soft)}. \tag{34}$$

The inner product between the TE_l aperture mode and the TE_m waveguide (a) mode is

$$\langle \tilde{\bar{e}}_l^h, \bar{e}_m^h \rangle_S = \iint_S \nabla_t \cdot \left[\Psi_m^{h*} \nabla_t \tilde{\Psi}_l^h \right] ds - \iint_S \Psi_m^{h*} \nabla_t^2 \tilde{\Psi}_l^h \, ds \tag{35}$$

where the scalar potentials satisfy the Helmholtz equation

$$\left[\nabla_t^2 + \left(\tilde{\kappa}_{cl}^h \right)^2 \right] \tilde{\Psi}_l^h = 0. \tag{36}$$

The two-dimensional divergence theorem, the directional derivative of a scalar, and boundary condition (33) for the aperture mode on C yield

$$\langle \tilde{\bar{e}}_l^h, \bar{e}_m^h \rangle_S = \left(\tilde{\kappa}_{cl}^h \right)^2 \iint_S \tilde{\Psi}_l^h \Psi_m^{h*} \, ds. \tag{37}$$

Similarly, the inner product over S of the TM aperture and waveguide modes is

$$\langle \tilde{\bar{e}}_l^e, \bar{e}_m^e \rangle_S = (\kappa_{cm}^e)^2 \int\int_S \tilde{\Psi}_l^e \Psi_m^{e*}\, ds. \tag{38}$$

The inner product of the TE_l aperture mode and the TM_m waveguide (a) mode is

$$\langle \tilde{\bar{e}}_l^h, \bar{e}_m^e \rangle_S = \int\int_S (\nabla_t \Psi_m^{e*} \times \nabla_t \tilde{\Psi}_l^h) \cdot d\bar{s} \tag{39}$$

which in view of vector identities and Stokes' theorem reduces to

$$\langle \tilde{\bar{e}}_l^h, \bar{e}_m^e \rangle_S = \oint_C \Psi_m^{e*} \nabla_t \tilde{\Psi}_l^h \cdot d\bar{l}. \tag{40}$$

Similarly, the inner product between the TM_l aperture mode and the TE_m waveguide (a) mode is trivial:

$$\langle \tilde{\bar{e}}_l^e, \bar{e}_m^h \rangle_S = \oint_C \tilde{\Psi}_l^e \nabla_t \Psi_m^{h*} \cdot d\bar{l} = 0 \tag{41}$$

from boundary condition (34) for the TM_l aperture mode on C.

V. Circular Iris

The circular iris of Fig. 1. is excited from the left by the TE_{11} mode with unity amplitude. The physical problem is symmetric in azimuthal angle ϕ and so the ϕ variation of the excitation is preserved. Only higher order modes of different radial or ρ variations are needed, i.e., only inclusion of the TE_{1r} and TM_{1r}, $r = 1, 2, \cdots$, modes in the waveguide and in the circular aperture is required. Henceforth, the "1" in the modal indices is dropped for notational convenience. The normalized scalar potentials for the natural modes of a circular waveguide of radius a with a single azimuthal variation are given by [6, pp. 66, 69]

$$\Psi_r^h = N_r^h J_1\left(\frac{x_{1r}'\rho}{a}\right)\cos\phi \qquad \Psi_r^e = N_r^e J_1\left(\frac{x_{1r}\rho}{a}\right)\sin\phi \tag{42}$$

where the normalization factors are

$$N_r^h = \sqrt{\frac{2}{\pi}}\frac{1}{\sqrt{x_{1r}'^2 - 1}\, J_1(x_{1r}')} \qquad N_r^e = \sqrt{\frac{2}{\pi}}\frac{1}{x_{1r} J_2(x_{1r})} \tag{43}$$

with x_{1r} and x_{1r}' the rth zeros of J_1 and J_1'. The transverse electric field modal vectors of (31)–(32) are

$$\bar{e}_r^h = \hat{\rho}\frac{N_r^h}{\rho} J_1\left(\frac{x_{1r}'\rho}{a}\right)\sin\phi + \hat{\phi}\frac{N_r^h x_{1r}'}{a} J_1'\left(\frac{x_{1r}'\rho}{a}\right)\cos\phi \tag{44}$$

$$\bar{e}_r^e = -\hat{\rho}\frac{N_r^e x_{1r}}{a} J_1'\left(\frac{x_{1r}\rho}{a}\right)\sin\phi - \hat{\phi}\frac{N_r^e}{\rho} J_1\left(\frac{x_{1r}\rho}{a}\right)\cos\phi \tag{45}$$

and the cutoff wavenumbers are

$$\kappa_{cr}^h = \frac{x_{1r}'}{a} \qquad \kappa_{cr}^e = \frac{x_{1r}}{a}. \tag{46}$$

The inner product over the circular aperture S between the TE_m waveguide (a) mode and the TE_k aperture mode, as given by (37), is

$$\begin{aligned}\langle \bar{e}_m^h, \tilde{\bar{e}}_k^h \rangle_S &= \left(\frac{x_{1k}'}{b}\right)^2 N_m^h \tilde{N}_k^h \int_0^{2\pi} \cos^2\phi\, d\phi \cdot \int_0^b J_1\left(\frac{x_{1m}'\rho}{a}\right) J_1\left(\frac{x_{1k}'\rho}{b}\right)\rho\, d\rho \\ &= \frac{2}{ab}\frac{J_1'\left(x_{1m}'\frac{b}{a}\right)}{J_1(x_{1m}')}\frac{x_{1m}' x_{1k}'^2}{\sqrt{(x_{1m}'^2 - 1)(x_{1k}'^2 - 1)}} \cdot \frac{1}{\left(\frac{x_{1k}'}{b}\right)^2 - \left(\frac{x_{1m}'}{a}\right)^2}, \qquad \frac{x_{1m}'}{a} \neq \frac{x_{1k}'}{b}.\end{aligned} \tag{47}$$

Similarly, (38) and (40) give

$$\langle \bar{e}_m^e, \tilde{\bar{e}}_k^e \rangle_S = \frac{2x_{1m}}{a^2}\frac{J_1\left(x_{1m}\frac{b}{a}\right)}{J_2(x_{1m})}\frac{1}{\left(\frac{x_{1k}}{b}\right)^2 - \left(\frac{x_{1m}}{a}\right)^2}, \qquad \frac{x_{1m}}{a} \neq \frac{x_{1k}}{b} \tag{48}$$

$$\langle \bar{e}_m^e, \tilde{\bar{e}}_k^h \rangle_S = \frac{-2J_1\left(x_{1m}\frac{b}{a}\right)}{x_{1m}\sqrt{x_{1k}'^2 - 1}\, J_2(x_{1m})}. \tag{49}$$

The modal admittances of the waveguide (a) TE_m and TM_m modes are

$$y_m^h = \begin{cases} \dfrac{\sqrt{(\kappa a)^2 - x_{1m}'^2}}{\eta\kappa a}, & \kappa a > x_{1m}' \\ \dfrac{\sqrt{x_{1m}'^2 - (\kappa a)^2}}{j\eta\kappa a}, & \kappa a < x_{1m}' \end{cases} \tag{50}$$

$$y_m^e = \begin{cases} \dfrac{\kappa a}{\eta\sqrt{(\kappa a)^2 - x_{1m}^2}}, & \kappa a > x_{1m} \\ \dfrac{j\kappa a}{\eta\sqrt{x_{1m}^2 - (\kappa a)^2}}, & \kappa a < x_{1m} \end{cases} \tag{51}$$

where $\kappa = \omega\sqrt{\mu\epsilon}$ and $\eta = \sqrt{\mu/\epsilon}$ are the intrinsic wavenumber and impedance, respectively, of the lossless dielectric. The left and right side matrices $\bar{Y}^a$ and $\bar{Y}^b$ are equal since waveguides (a) and (b) are identical. Henceforth, the superscript a is dropped from the admittance matrix and the factor 2 is dropped from the forcing term (29). The three types of symmetric matrix elements are

$$Y_{kl}^{hh} = \sum_{m=1}^{\infty} y_m^h \langle \bar{e}_m^h, \tilde{\bar{e}}_k^h \rangle_S \langle \tilde{\bar{e}}_l^h, \bar{e}_m^h \rangle_S + \sum_{m=1}^{\infty} y_m^e \langle \bar{e}_m^e, \tilde{\bar{e}}_k^h \rangle_S \langle \tilde{\bar{e}}_l^h, \bar{e}_m^e \rangle_S \tag{52}$$

$$Y_{kl}^{ee} = \sum_{m=1}^{\infty} y_m^e \langle \bar{e}_m^e, \tilde{\bar{e}}_k^e \rangle_S \langle \tilde{\bar{e}}_l^e, \bar{e}_m^e \rangle_S \tag{53}$$

$$Y_{kl}^{he} = \sum_{m=1}^{\infty} y_m^e \langle \bar{e}_m^e, \tilde{\bar{e}}_k^h \rangle_S \langle \tilde{\bar{e}}_l^e, \bar{e}_m^e \rangle_S. \tag{54}$$

If all TM_{1m} and TE_{1m} modes except the TE_{11} mode are cut off in the waveguide, then (52) is

$$Y_{kl}^{hh} = G_{kl}^{hh} + j\left(B_{kl}^{hh} + C_{kl}^{hh}\right) \tag{55}$$

where the real part is

$$G_{kl}^{hh} = \frac{4\left(\frac{b}{a}\right)^2 \sqrt{(\kappa a)^2 - x_{11}'^2}\, x_{11}'^2 J_1'^2\left(\frac{b}{a}x_{11}'\right) x_{1k}'^2 x_{1l}'^2}{\eta \kappa a J_1^2(x_{11}')\left(x_{11}'^2 - 1\right)\sqrt{\left(x_{1k}'^2 - 1\right)\left(x_{1l}'^2 - 1\right)}} \left\{\left[x_{1k}'^2 - \left(\frac{b}{a}x_{11}'\right)^2\right]\left[x_{1l}'^2 - \left(\frac{b}{a}x_{11}'\right)^2\right]\right\}^{-1}. \tag{56}$$

Using the principal asymptotic forms of the Bessel functions [21, pp. 364, 371]

$$\begin{aligned} J_\nu(x) &\sim \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) \qquad x_{\nu m} \sim \left(m + \frac{\nu}{2} - \frac{1}{4}\right)\pi \\ J_\nu'(x) &\sim \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\nu\pi}{2} + \frac{\pi}{4}\right) \qquad x_{\nu m}' \sim \left(m + \frac{\nu}{2} + \frac{1}{4}\right)\pi \end{aligned} \tag{57}$$

a Kummer transform [22, p. 203] on the slowly converging series of the susceptive parts yields

$$\begin{aligned} B_{kl}^{hh} = \frac{-4\left(\frac{a}{b}\right)^2 x_{1k}'^2 x_{1l}'^2}{\eta\kappa a\sqrt{\left(x_{1k}'^2 - 1\right)\left(x_{1l}'^2 - 1\right)}} & \left[\sum_{m=2}^{\infty}\left\{\frac{x_{1m}'^2\sqrt{x_{1m}'^2 - (\kappa a)^2}\, J_1'^2\left(\frac{b}{a}x_{1m}'\right)}{J_1^2(x_{1m}')\left(x_{1m}'^2 - 1\right)\left[x_{1m}'^2 - \left(\frac{a}{b}x_{1k}'\right)^2\right]\left[x_{1m}'^2 - \left(\frac{a}{b}x_{1l}'\right)^2\right]} - \frac{\cos^2\left[\frac{b}{a}\pi\left(m + \frac{3}{4}\right) - \frac{\pi}{4}\right]}{\frac{b}{a}(\pi m)^3}\right\}\right. \\ & + \frac{1}{2\pi^3\frac{b}{a}}\left\{\sum_{m=1}^{\infty}\frac{1}{m^3} + \cos\left(\frac{3b\pi}{2a}\right)\sum_{m=1}^{\infty}\frac{\sin\left(\frac{2b\pi}{a}m\right)}{m^3}\right. \\ & \left.\left. + \sin\left(\frac{3b\pi}{2a}\right)\sum_{m=1}^{\infty}\frac{\cos\left(\frac{2b\pi}{a}m\right)}{m^3} - 1 - \sin\left(\frac{7b\pi}{2a}\right)\right\}\right] \end{aligned} \tag{58}$$

$$\begin{aligned} C_{kl}^{hh} = \frac{4\kappa a}{\eta\sqrt{\left(x_{1k}'^2 - 1\right)\left(x_{1l}'^2 - 1\right)}} & \left[\sum_{m=1}^{\infty}\left\{\frac{J_1^2\left(\frac{b}{a}x_{1m}\right)}{x_{1m}^2 J_2^2(x_{1m})\sqrt{x_{1m}^2 - (\kappa a)^2}} - \frac{\cos^2\left[\frac{b}{a}\pi\left(m + \frac{1}{4}\right) - \frac{3\pi}{4}\right]}{\frac{b}{a}(\pi m)^3}\right\}\right. \\ & \left. + \frac{1}{2\pi^3\frac{b}{a}}\left\{\sum_{m=1}^{\infty}\frac{1}{m^3} - \cos\left(\frac{b\pi}{2a}\right)\sum_{m=1}^{\infty}\frac{\sin\left(\frac{2b\pi}{a}m\right)}{m^3} - \sin\left(\frac{b\pi}{2a}\right)\sum_{m=1}^{\infty}\frac{\cos\left(\frac{2b\pi}{a}m\right)}{m^3}\right\}\right]. \end{aligned} \tag{59}$$

Similarly, the other aperture admittance matrix elements are accelerated as

$$\begin{aligned} Y_{kl}^{ee} = \frac{j4\kappa a}{\eta} & \left[\sum_{m=1}^{\infty}\left\{\frac{x_{1m}^2 J_1^2\left(\frac{b}{a}x_{1m}\right)}{J_2^2(x_{1m})\sqrt{x_{1m}^2 - (\kappa a)^2}\left[x_{1m}^2 - \left(\frac{a}{b}x_{1k}\right)^2\right]\left[x_{1m}^2 - \left(\frac{a}{b}x_{1l}\right)^2\right]} - \frac{\cos^2\left[\frac{b}{a}\pi\left(m + \frac{1}{4}\right) - \frac{3\pi}{4}\right]}{\frac{b}{a}(\pi m)^3}\right\}\right. \\ & \left. + \frac{1}{2\pi^3\frac{b}{a}}\left\{\sum_{m=1}^{\infty}\frac{1}{m^3} - \cos\left(\frac{b\pi}{2a}\right)\sum_{m=1}^{\infty}\frac{\sin\left(\frac{2b\pi}{a}m\right)}{m^3} - \sin\left(\frac{b\pi}{2a}\right)\sum_{m=1}^{\infty}\frac{\cos\left(\frac{2b\pi}{a}m\right)}{m^3}\right\}\right] \end{aligned} \tag{60}$$

$$Y_{kl}^{he} = \frac{j4\kappa a}{\eta\sqrt{x_{1k}'^2 - 1}}\left[\sum_{m=1}^{\infty}\left\{\frac{J_1^2\left(\frac{b}{a}x_{1m}\right)}{J_2^2(x_{1m})\sqrt{x_{1m}^2-(\kappa a)^2}\left[x_{1m}^2-\left(\frac{b}{a}x_{1l}\right)^2\right]} - \frac{\cos^2\left[\frac{b}{a}\pi\left(m+\frac{1}{4}\right)-\frac{3\pi}{4}\right]}{\frac{b}{a}(\pi m)^3}\right\}\right.$$

$$\left. + \frac{1}{2\pi^3\frac{b}{a}}\left\{\sum_{m=1}^{\infty}\frac{1}{m^3} - \cos\left(\frac{b\pi}{2a}\right)\sum_{m=1}^{\infty}\frac{\sin\left(\frac{2b\pi}{a}m\right)}{m^3} - \sin\left(\frac{b\pi}{2a}\right)\sum_{m=1}^{\infty}\frac{\cos\left(\frac{2b\pi}{a}m\right)}{m^3}\right\}\right]. \tag{61}$$

The asymptotic series above are of the form [23, p. 579]

$$\sum_{m=1}^{\infty}\frac{\sin(mx)}{m^3} = \frac{\pi^2 x}{6} - \frac{\pi x^2}{4} + \frac{x^3}{12}, \qquad 0 < x < 2\pi$$

$$\sum_{m=1}^{\infty}\frac{\cos(mx)}{m^3} \approx \zeta(3) + \frac{x^2}{2}\left(\ln x - \frac{3}{2}\right) - \frac{x^4}{288} - \frac{x^6}{86{,}400}, \qquad 0 < x < 2\pi \tag{62}$$

where the Riemann zeta function is [21, p. 811]

$$\sum_{m=1}^{\infty}\frac{1}{m^3} = \zeta(3) = 1.2020569. \tag{63}$$

If the excitation is the TE_1 waveguide mode of unity amplitude, then all TM current elements (29) are zero by (41) and the TE current elements are

$$I_k^h = \frac{2\frac{b}{a}\sqrt{(\kappa a)^2 - x_{11}'^2}\, x_{1k}'^2 x_{11}' J_1'\left(\frac{b}{a}x_{11}'\right)}{\eta\kappa a\sqrt{(x_{11}'^2-1)(x_{1k}'^2-1)}\left[x_{1k}'^2 - \left(\frac{b}{a}x_{11}'\right)^2\right]J_1(x_{11}')}. \tag{64}$$

Using an aperture basis of M TE aperture modes and N TM aperture modes, i.e.,

$$\bar{E}_{\text{aper}} = \sum_{l=1}^{M}\tilde{V}_l^h \tilde{\bar{e}}_l^h + \sum_{l=1}^{N}\tilde{V}_l^e \tilde{\bar{e}}_l^e \tag{65}$$

for (25), the matrix equation (26) is

$$\begin{bmatrix} \bar{Y}^{hh} & \bar{Y}^{he} \\ \bar{Y}^{eh} & \bar{Y}^{ee} \end{bmatrix}\begin{bmatrix} \tilde{\bar{V}}^h \\ \tilde{\bar{V}}^e \end{bmatrix} = \begin{bmatrix} \bar{I}^h \\ \bar{0} \end{bmatrix}. \tag{66}$$

VI. Results

Fig. 3 is a computer-generated plot of the resultant aperture electric field direction using ten TE and ten TM aperture modes for an iris of size $b/a = 0.5$ at a frequency corresponding to $\kappa a = 2.5$. This transverse electric field pattern appears to be a perturbed TE_{11} mode. The next largest TE aperture mode is the TE_{12} mode, which is 17.6 dB below the TE_{11} mode in this aperture. The largest TM aperture mode is the TM_{11} mode, which is 9.6 dB below the TE_{11} aperture mode. The resultant numerical amplitudes of the higher order TE_{1n} and TM_{1n} aperture modes decay approximately as $n^{-1.6}$ and $n^{-0.6}$, respectively. The asymptotic edge conditions [24, p. 4] at the aperture boundary ($\rho \sim b$)

$$E_\rho \sim (1-\rho/b)^{-1/2} \qquad E_\phi \sim (1-\rho/b)^{1/2} \tag{67}$$

indicate that the higher order TE_{1n} and TM_{1n} aperture modes decay at least as fast as $n^{-1.5}$ and $n^{-0.5}$, respectively, using the method of Carslaw [25, p. 274] where the approximation (65) is recognized as a finite Fourier–Bessel series representation of the actual aperture field. The singularity in the radial aperture electric field is responsible for the nonuniform convergence and associated Gibb's phenomenon. Fig. 4. depicts the magnitude of the radial electric field $|E_\rho|$ in the $\phi = \pi/2$ plane and of the azimuthal electric field $|E_\phi|$ in the $\phi = 0$ plane, using the ten-TE and ten-TM aperture mode approximation, corresponding to Fig. 3. Fig. 5 uses an aperture basis of 20 TE and 20 TM modes, which gives rise to a higher value of $|E_\rho|$ at the aperture edge.

The effect of the iris on the incident TE_{11} mode of the waveguide can be characterized by an equivalent shunt susceptance B, normalized with respect to the characteristic wave admittance of the waveguide TE_{11} mode. The numerical convergence of B for a 50-percent iris ($b/a = 0.5$) operated 36 percent above cutoff ($\kappa a = 2.5$) is demonstrated in Table I. The variation of susceptance with iris size and frequency is illustrated in Fig. 6. The value $\kappa a = 1.9$ corresponds to a frequency slightly above TE_{11} cutoff. Note that, for some values of κa, a resonant or capacitive iris is possible. In contrast to the resonant iris for rectangular waveguides [26, p. 170], the resonant iris and the capacitive iris are possible only over a restricted range of frequencies for the geometry of Fig. 1. The resonant iris is, however, obtainable for all values of b/a. Fig. 7 shows a comparison of the Galerkin results with the Bethe small-hole theory results of Marcuvitz [6, p. 243]. A logarithmic scale is used to permit a more precise comparison. The results agree only for apertures small compared with wavelength; thus the results near cutoff agree for relatively large b/a. As κa increases, the Bethe small-hole results are accurate only for smaller relative b/a, as expected. Finally, although experimental results cannot be obtained for the zero-thickness iris, recent experimental results for very thin irises [18], [19] show excellent agreement with the Galerkin results.

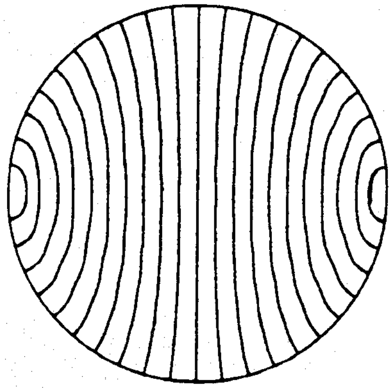

Fig. 3. Aperture electric field direction using 20 aperture modes ($M = N = 10$) with $\kappa a = 2.5$ and $b/a = 0.5$.

TABLE I
CONVERGENCE OF EQUIVALENT SHUNT SUSCEPTANCE FOR $\kappa a = 2.5$ AND $b/a = 0.5$

Number of Aperture Modes		Normalized Shunt Susceptance, B	CPU Time on Vax 8650 (sec)
M (TE)	N (TM)		
1	1	-3.526	2.1
5	5	-2.894	2.6
10	10	-2.809	4.1
20	20	-2.767	20.4
40	40	-2.745	80.8

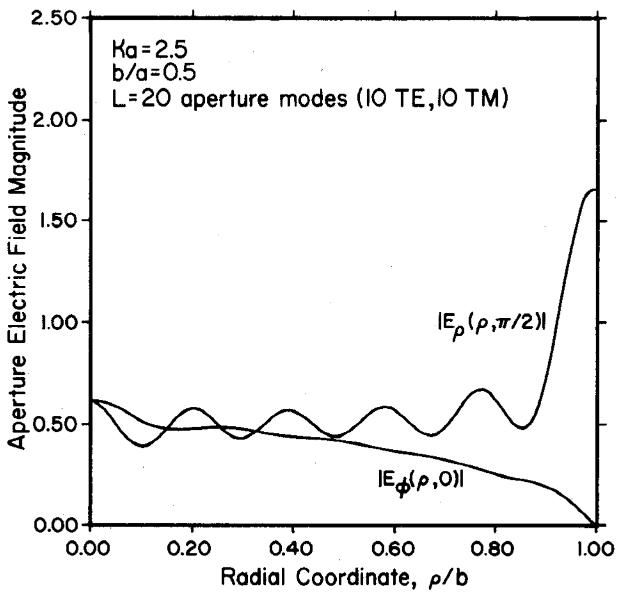

Fig. 4. Principal plane aperture electric field magnitude using a basis of 20 aperture modes.

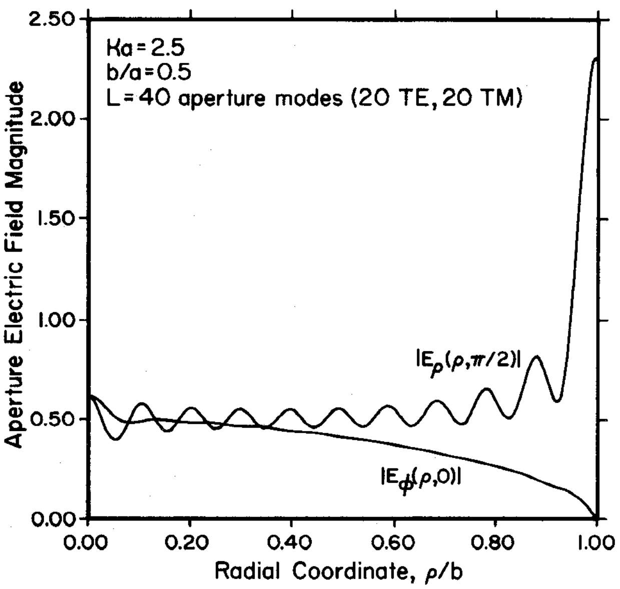

Fig. 5. Principal plane aperture electric field magnitude using a basis of 40 aperture modes.

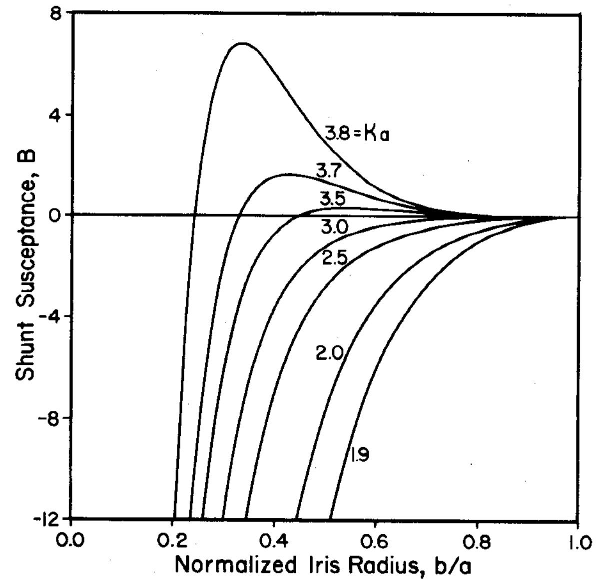

Fig. 6. Shunt susceptance of circular iris in circular waveguide.

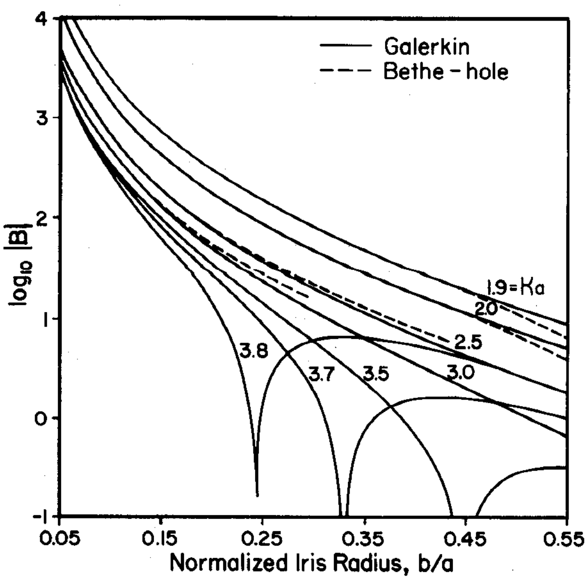

Fig. 7. Shunt susceptance of circular iris in circular waveguide (logarithmic plot): comparison with Bethe small-hole theory.

VII. Conclusions

The thin circular iris in circular waveguide has been treated by a Galerkin moment method to obtain the aperture field and shunt susceptance. A general formulation, which is valid for arbitrary modes incident from left and right, is obtained and then specialized to consider TE_{11}-mode excitation. The convergence of the infinite series for the aperture admittance matrix elements is accelerated by the Kummer transform. Computations for shunt susceptance indicate that the iris may be resonant or capacitive over a restricted frequency range. Small aperture data agree with Bethe small-hole theory.

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References

[1] L. S. Sheingold, "The susceptance of a circular obstacle to an incident dominant circular-electric wave," *J. Appl. Phys.*, vol. 24, no. 4, pp. 414–422, Apr. 1953.
[2] V. M. Papadopoulos, "The scattering effect of a junction between two circular waveguides," *Quart. J. Mech. Appl. Math.*, vol. 10, pt. 2, pp. 191–209, 1957.
[3] Yu. V. Vaisleib, G. N. Zelenskii, and L. N. Kulikov, "Scattering of electromagnetic waves at abrupt discontinuity in the transverse cross section of round waveguides," *Radio Eng. Electron. Phys.*, vol. 20, no. 1, p. 39–48, Jan. 1975.
[4] T. Sugiura and H. Suga, "The susceptance of an annular metallic strip in a circular waveguide with incident TE_{01} mode," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-27, pp. 160–167, Feb. 1979.
[5] A. A. Kirilenko and N. P. Yashina, "Rigorous mathematical analysis and electrodynamic characteristics of a diaphragm in a circular waveguide," *Radio Phys. Quantum Electron.*, vol. 23, no. 11, pp. 897–903, Nov. 1980.
[6] N. Marcuvitz, *Waveguide Handbook* (MIT Rad Lab Series, vol. 10). New York: McGraw-Hill, 1951.
[7] D. S. Gubskii, V. P. Lyapin, and G. P. Sinyavskii, "Electrodynamic calculation of the parameters of a diaphragmed junction between round waveguides," *Radio Eng. Electron. Phys.*, vol. 29, no. 1, pp. 12–19, Jan. 1984.
[8] R. F. Harrington and J. R. Mautz, "A generalized network formulation for aperture problems," *IEEE Trans. Antennas Propagat.*, vol. AP-24, no. 6, pp. 870–873, Nov. 1976.
[9] H. Auda and R. F. Harrington, "A moment solution for waveguide junction problems," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-31, pp. 512–520, July 1983.
[10] A. Wexler, "Solution of waveguide discontinuities by modal analysis," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-15, pp. 508–517, Sept. 1967.
[11] J. D. Wade and R. H. MacPhie, "Scattering at circular-to-rectangular waveguide junctions," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-34, pp. 1085–1091, Nov. 1986.
[12] S. G. Mikhlin, *Variational Methods in Mathematical Physics*. New York: Pergamon, 1964.
[13] A. M. Lerer, V. P. Lyapin, and G. P. Sinyavskii, "Displacement of rectangular waveguides," *Radio Phys. Quantum Electron.*, vol. 25, no. 8, pp. 671–678, Aug. 1982.
[14] V. P. Lyapin, M. B. Manuilov, and G. P. Sinyavskii, "Calculation of diaphragms and dielectric windows in multimode rectangular waveguides," *Sov. J. Commun. Tech. Electron.*, vol. 30, no. 4, pp. 130–138, Apr. 1985.
[15] J. D. Hunter, "The displaced rectangular waveguide junction and its use as an adjustable reference reflection," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-32, pp. 387–394, Apr. 1984.
[16] R. Mittra, T. Itoh, and T. Li, "Analytical and numerical studies of the relative convergence phenomenon arising in the solution of an integral equation by the moment method," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-20, pp. 96–104, Feb. 1972.
[17] V. P. Lyapin, V. S. Mikhalevsky, and G. P. Sinyavskii, "Taking into account the edge condition in the problem of diffraction waves on step discontinuity in plate waveguide," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-30, pp. 1107–1109, July 1982.
[18] R. W. Scharstein, "Electromagnetic analysis of waveguide junctions and irises using Galerkin's method," Ph.D. dissertation, Syracuse University, Syracuse, NY, 1986.
[19] R. W. Scharstein and A. T. Adams, "Galerkin solution for the thin and thick circular iris in circular waveguide," in *Proc. 1987 Nat. Radio Sci. Meeting* (Boulder, CO), Jan. 11–15, 1987, p. 169.
[20] I. Stakgold, *Green's Functions and Boundary Value Problems*. New York: Wiley, 1979.
[21] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions*. Washington, DC: Gov. Printing Off., 1972.
[22] B. P. Demidovich and I. A. Maron, *Computational Mathematics*. Moscow: MIR Publishers, 1981.
[23] R. E. Collin, *Field Theory of Guided Waves*. New York: McGraw-Hill, 1960.
[24] R. Mittra and S. W. Lee, *Analytical Techniques in the Theory of Guided Waves*. New York: MacMillan, 1971.
[25] H. S. Carslaw, *An Introduction to the Theory of Fourier's Series and Integrals*, 3rd ed. New York: Dover, 1950.
[26] C. G. Montgomery, R. H. Dicke, and E. M. Purcell, *Principles of Microwave Circuits* (MIT Rad Lab Series, vol. 8). New York: McGraw-Hill, 1948.

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